

# Renormalization Group Derivation of Phase Equations

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## Abstract

Phase equations describing the evolution of large scale modulation of spatially periodic patterns in two dimensional systems are derived by employing the renormalization group method. A general formula for phase diffusion coefficients is given under certain conditions.

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## I. INTRODUCTION

Extracting a simpler representation of a dynamical system by restricting what we wish to describe, which is called system reduction, makes us possible to understand an apparently complicated dynamical phenomenon in a simple way. In particular, perturbative system reduction is formulated when we can express the system in question by introducing a small parameter. The idea of system reduction may have a long history since Boltzmann discussed a transport equation for systems consisting of many gas molecules. In the last three decades, systematic methods for perturbative system reduction have been developed and applied to many examples of dynamical phenomena appearing especially in fluid systems and reaction diffusion systems [1–3].

However, those who are not familiar with the methods often claim that they seem to include a sort of art. Most artistic procedure may be the introduction of multiple-scales variables [4]. Actually, the choice of a set of scaled variables in the multiple-scales expansion requires a guess of the final result and is justified post hoc. Thus, one may wish to have a standard method for perturbative system reduction without any post hoc justification. Such a method should be completely mechanical in a sense that our intuition is excluded out.

Here, we should remark that the multiple-scales technique was developed rather recently in a history of perturbation theory on differential equations. It is difficult to specify the origin of the perturbation theory, but Poincaré was the first to discuss this subject systematically [5]. The problem was to avoid secular terms when a perturbative expansion is considered. By secular term is meant a term unbounded with time. Subsequently, Krylov and Bogolyubov proposed a practically convenient and rigorously justified mathematical method to study nonlinear oscillations [6]. The essence of these early important works is in the notion of a normal form of evolution equations. That is, by making use of a change of variables, evolution equations can be transformed to a simpler form. This notion has no relation to multiple-scales. Thus, one may develop a mechanical formulation relying on the idea of a normal form. In fact, such perturbation theory has been discussed by Bogachevsky and

Povzner [7]. They developed a formalism generalizing the Poincaré-Bogolyubov-Krynov-Mitropolsky notion of a normal form. Kuramoto also attempted to formulate a perturbative system reduction with a geometrical interpretation based on the similar idea [8]. There seems to be no procedure of guessing a final form in their formulation. A subtle point is that the existence of a well-defined evolution equation for new variables introduced by a change of variables must be assumed. Precisely speaking, it may be regarded as post hoc justification.

Recently, a highly mechanical formulation for asymptotic perturbation theory including the perturbative system reduction has been proposed by Goldenfeld, Oono and their collaborators [9–11]. This was called the renormalization group (RG) approach. In their formulation, a naive perturbative expansion is a starting point in contrast to many theories which reorganize a perturbation series so that secular terms do not appear. The secular divergence is renormalized and the RG analysis is employed. These procedures are well-known in field theory. The RG approach has been applied successfully to many examples. In the context of the perturbative system reduction, it is worthwhile noting that the obtained evolution equation through the system reduction is just the RG equation. Actually, recently, the Boltzmann equation [12] and an Euclid-invariant amplitude equation [13] have been derived as RG equations. One may expect that universal equations in some sense will be regarded as renormalization group equations.

This paper will give a new example supporting this conjecture. We will derive, by employing the RG method, nonlinear phase equations describing the evolution of large scale modulation of spatially periodic patterns in two dimensional systems. A phase equation for spatially periodic patterns was first derived by Pomeau and Manneville [14]. Subsequently, phase equations for patterns have been derived in various contexts and extended to nonlinear ones [1]. Practically, a method developed by Cross and Newell seems most efficient to derive phase equations [15]. Cross and Newell also proposed a canonical form of nonlinear phase equations. The phase equation we will derive is equivalent to a Cross-Newell equation.

The outline of this paper is as follows. In section II, we introduce model equations

to be analyzed. We analyze rather general partial differential equations including a Swift-Hohenberg (SH) equation [16]. In section III, a naive perturbation will be performed. Complicated calculation of secular terms will be shown in the Appendix A. In section IV, we will renormalize the secular terms and obtain a perturbation series uniformly valid. In section V, nonlinear phase equations will be obtained by the renormalization group analysis, and a general formula for phase diffusion coefficients will be given. In Appendix B, phase diffusion coefficients for the SH eq. will be calculated. Section VI will be devoted to discussion.

## II. MODEL

We consider partial differential equations (PDEs) of the form

$$\frac{\partial w}{\partial t} = F(\{\frac{\partial^\lambda}{\partial x_1^\lambda} \frac{\partial^\mu}{\partial x_2^\mu} w\}_{(0 \leq \lambda + \mu \leq M)}), \quad (1)$$

which describes the evolution to spatially periodic patterns in a two dimensional space.  $w$  denotes a scalar field, and the generalization to the multi-component cases is straightforward.  $F$  is a polynomial of partial derivatives of  $w$  up to the  $M$ -th order. This implies that we do not consider an equation with a nonlocal coupling. The assumption for  $F$  will be stated in order. First,  $F$  is assumed to be symmetric with respect to a parity transformation  $\vec{x} \rightarrow -\vec{x}$ , while the rotational invariance is not assumed. Second, as known in many cases, it is assumed that the spatially periodic solutions form a family expressed by

$$w(\vec{x}) = f(\theta, \vec{k}), \quad (2)$$

$$\theta = \vec{k}\vec{x} + \phi, \quad (3)$$

where  $f$  is a  $2\pi$  periodic function in  $\theta$ ,  $\vec{k}$  is an arbitrary constant vector in a certain range and  $\phi$  is an arbitrary constant phase. Further, for simplicity, we assume that  $f$  can be chosen as a parity symmetric one, that is,

$$f(\theta, \vec{k}) = f(-\theta, \vec{k}). \quad (4)$$

We can easily generalize our argument to the case that this assumption does not hold, but the analysis becomes complicated.

We pay attention to a late stage in the pattern formation process, and we assume that almost solutions to the equation approach to a neighborhood of a family of solutions expressed by Eq.(2) as time goes on. Then, since the solutions satisfy  $F = 0$ ,  $\partial w / \partial t$  may be regarded as a perturbation. Physically, this expectation is reasonable because the evolution of patterns becomes slower on a later stage in the pattern formation process. In order to formulate a perturbation problem, we rewrite Eq.(1) as

$$\epsilon \frac{\partial w}{\partial t} = F(\{\frac{\partial^\lambda}{\partial x_1^\lambda} \frac{\partial^\mu}{\partial x_2^\mu} w\}_{(0 \leq \lambda + \mu \leq M)}). \quad (5)$$

Do not confuse this procedure with a multiple scale expansion method. We have just set up the system which will be analyzed, by restricting what we wish to describe. However, someone may claim that the above procedure is not mechanical and too formal because  $\epsilon$  is not an observable parameter. We agree the former claim and will discuss it in the final section. Also, the latter claim may be true in the sense that we cannot say the value of  $\epsilon$ . We consider this equation under the asymptotic condition  $\epsilon \rightarrow 0_+$ .

The simplest example of Eq.(5) is a Swift-Hohenberg (SH) equation:

$$\epsilon \partial_t w = R w - w^3 + (1 - (1 + \Delta)^2) w, \quad (6)$$

where  $R$  is a control parameter. The spatially periodic solutions of the SH equation are expressed by Eq.(2) with Eq.(3) and satisfy the condition Eq.(4). (See Appendix B.) Note that  $M = 4$  for the SH eq. We will develop an argument applicable to Eq.(5). One can consider the SH eq. as a concrete model.

### III. NAIVE PERTURBATION

We employ a naive perturbative expansion in  $\epsilon$ . The solution is expanded as

$$w = w_0(x, t) + \epsilon w_1(x, t) + \cdots. \quad (7)$$

The 0-th order solution is given by

$$w_0(\vec{x}, t) = f(\theta, \vec{k}(t)) = f(\vec{k}(t)\vec{x} + \phi(t), \vec{k}(t)), \quad (8)$$

where  $\phi(t)$  and  $\vec{k}(t)$  are arbitrary functions in  $t$ . Proceeding the first order in  $\epsilon$ , we obtain

$$\frac{\partial \phi}{\partial t} \frac{\partial f}{\partial \theta} + \frac{\partial \vec{k}}{\partial t} (\vec{x} \frac{\partial f}{\partial \theta} + \frac{\partial f}{\partial \vec{k}}) = \hat{L}^{(0)} w_1. \quad (9)$$

Here,  $\hat{L}^{(0)}$  is a linearized operator around the solution and formally expressed by

$$\hat{L}^{(0)} = \left. \frac{\delta F}{\delta w} \right|_{w=f}, \quad (10)$$

where  $\delta/\delta w$  denotes the functional derivative along  $w$ .  $\hat{L}^{(0)}$  is symmetric with respect to the transformation  $\vec{x} \rightarrow -\vec{x}$  and includes partial derivative operators up to the  $M$ -th order.

Now, we solve Eq.(9). Mathematically speaking, we should specify an appropriate functional space for the solutions, but we do not enter the mathematical details. Nevertheless, noting the  $\vec{x}$  dependence in the left-handed side of Eq.(9), we can expect naively that the general solutions take a form

$$w_1 = \sum_{n=0}^{M+1} x_{(1:n)} a_{(1:n)}^{(n)}, \quad (11)$$

where we have introduced an abbreviation

$$x_{(1:n)} = x_{i_1} x_{i_2} \cdots x_{i_n}, \quad (12)$$

$$a_{(1:n)}^{(n)} = a_{i_1 i_2 \cdots i_n}^{(n)}. \quad (13)$$

We will use a similar abbreviation for tensors with indices  $i_k \cdots i_m$ . (Also,  $a_{(1:0)}^{(0)}$  means a scalar variable  $a^{(0)}$  and  $x_{(1:0)} = 1$ .) The tensor  $a_{(1:n)}^{(n)}$  is a function in  $\theta (= \vec{k}(t)\vec{x} + \phi(t))$  and in  $t$  (through  $t$  dependence of  $\partial \phi / \partial t$  and  $\partial \vec{k} / \partial t$ ) and can be assumed to be symmetric with respect to the permutation of indices.

Here, in order to evaluate  $\hat{L}^{(0)} w_1$ , we define the  $n$ -th order tensor operator  $\hat{L}^{(n)}$  recursively by the formula

$$\hat{L}_{(1:n-1)}^{(n-1)} [x_{i_n} g(\vec{x})] = x_{i_n} \hat{L}_{(1:n-1)}^{(n-1)} g(\vec{x}) + \hat{L}_{(1:n)}^{(n)} g(\vec{x}), \quad (14)$$

where  $g(\vec{x})$  is an arbitrary function. As a concrete example,  $\hat{L}^{(0)}$ ,  $\hat{L}^{(1)}$  and  $\hat{L}^{(2)}$  for the SH eq. are given by

$$\hat{L}^{(0)} = R - 3w_0^2 - (1 + \Delta)^2, \quad (15)$$

$$\hat{L}_j^{(1)} = -4(1 + \Delta)\partial_j, \quad (16)$$

$$\hat{L}_{ij}^{(2)} = -4(1 + \Delta)\delta_{ij} - 8\partial_i\partial_j. \quad (17)$$

Then, by using the operators  $\{\hat{L}^{(n)}\}_{n=0}^{M+1}$ ,  $\hat{L}^{(0)}w_1$  is written as

$$\hat{L}^{(0)}w_1 = \sum_{k=0}^{M+1} x_{(1:k)} \sum_{n=k}^{M+1} {}_nC_k \hat{L}_{(k+1:n)}^{(n-k)} a_{(1:n)}^{(n)}. \quad (18)$$

Substituting this equation to Eq.(9), we obtain

$$\sum_{k=0}^{M+1} x_{(1:k)} b_{(1:k)}^{(k)} = 0, \quad (19)$$

where

$$b_{(1:k)}^{(k)} = \sum_{n=k}^{M+1} {}_nC_k \hat{L}_{(k+1:n)}^{(n-k)} a_{(1:n)}^{(n)}, \quad (20)$$

for  $k \geq 2$ , and

$$b_{(1:1)}^{(1)} = \sum_{n=1}^{M+1} n \hat{L}_{(2:n)}^{(n-1)} a_{(1:n)}^{(n)} - \frac{\partial k_{i_1}}{\partial t} \frac{\partial f}{\partial \theta}, \quad (21)$$

$$b_{(1:0)}^{(0)} = \sum_{n=0}^{M+1} \hat{L}_{(1:n)}^{(n)} a_{(1:n)}^{(n)} - \frac{\partial k_l}{\partial t} \frac{\partial f}{\partial k_l} - \frac{\partial \phi}{\partial t} \frac{\partial f}{\partial \theta}. \quad (22)$$

Since  $\vec{x}$  is arbitrary and  $b^{(n)}$  is a periodic function in  $\vec{x}$ , Eq.(19) leads  $M + 2$  equations

$$b_{(1:k)}^{(k)} = 0, \quad (0 \leq k \leq M + 1). \quad (23)$$

We can determine the tensors  $\{a^{(n)}\}$  by solving the set of equations and obtain

$$w_1 = (x_{lmn}\rho_{lmn}^{(3)} + x_{lm}\rho_{lm}^{(2)} + x_l\rho_l^{(1)})\frac{\partial f}{\partial \theta} + (3x_{lm}\rho_{lmn}^{(3)} + 2x_l\rho_{ln}^{(2)})\frac{\partial f}{\partial k_n} + a^{(0)}. \quad (24)$$

Here,  $\rho_{lmn}^{(3)}$  and  $\rho_{lm}^{(2)}$  do not depend on  $\vec{x}$  and satisfy

$$6\rho_{lmn}^{(3)}D_{mn} = \frac{\partial k_l}{\partial t}, \quad (25)$$

$$2\rho_{lm}^{(2)}D_{lm} = \frac{\partial \phi}{\partial t}, \quad (26)$$

with

$$D_{mn} = \frac{1}{2}(\Phi_0^\dagger, \hat{L}_{mn}^{(2)}\Phi_0) + (\Phi_0^\dagger, \hat{L}_m^{(1)}\frac{\partial f}{\partial k_n}), \quad (27)$$

where  $(u, v)$  denotes the inner product in the space  $V_{per} = \{u|u \in L_2[0, 2\pi], \quad u(0) = u(2\pi)\}$ ,  $\Phi_0 = \partial f / \partial \theta$  is a null eigenvector, and  $\Phi_0^\dagger$  is the adjoint null vector normalized to  $(\Phi_0^\dagger, \Phi_0) = 1$ .  $\rho_l$  is an arbitrary constant and  $a^{(0)}$  is a periodic function with a certain arbitrariness. We do not need to know the explicit form of  $a^{(0)}$ . Eqs.(24) - (27) are derived in Appendix A.

#### IV. RENORMALIZATION

The naive perturbation series up to the first order in  $\epsilon$  is summarized as

$$w = f(\vec{k}(t)\vec{x} + \phi(t), \vec{k}(t)) + \epsilon w_1, \quad (28)$$

where  $w_1$  is given by Eq.(24). This perturbation result does not make a sense for large  $|\vec{x}|$  because of the singular behavior of  $w_1$ . However, this result becomes meaningful locally around an arbitrary position  $\vec{X}$  when we successfully separate the singular behavior and renormalize it to the arbitrariness in the zero-the order solution. This operation corresponds to the renormalization in field theory.

We now show that the singular behavior in  $w_1$  is renormalizable. First, we split  $x_{(1:n)}$  to  $x_{(1:n)} = x_{(1:n)} - X_{(1:n)} + X_{(1:n)}$ , where  $x_{(1:n)}^R = x_{(1:n)} - X_{(1:n)}$  is not large when we focus on the system around the position  $\vec{X}$ . Then, the singular part in  $w_1$ , denoted by  $w_1^{sing}$ , becomes

$$w_1^{sing} = (X_{lmn}\rho_{lmn}^{(3)} + X_{lm}\rho_{lm}^{(2)} + X_l\rho_l^{(1)})\frac{\partial f}{\partial \theta} + (3X_{lm}\rho_{lmn}^{(3)} + 2X_l\rho_{ln}^{(2)})\frac{\partial f}{\partial k_n}. \quad (29)$$

We wish to renormalize these singular terms to the arbitrariness in the 0-th order solution,  $f(\vec{k}\vec{x} + \phi, \vec{k})$ . Let us recall that  $\phi$  and  $\vec{k}$  in  $f$  are arbitrary functions in  $t$ . We then define  $\phi^R(\vec{X}, t)$  and  $\vec{k}^R(\vec{X}, t)$  by



$$\phi(t) = \phi^R(\vec{X}, t) + \epsilon \delta \phi(\vec{X}, t), \quad (30)$$

$$\vec{k}(t) = \vec{k}^R(\vec{X}, t) + \epsilon \delta \vec{k}(\vec{X}, t). \quad (31)$$

$\phi^R(\vec{X}, t)$  and  $\delta \phi(\vec{X}, t)$  are called a renormalized phase and a counter term, respectively.  $\vec{k}^R$  and  $\delta \vec{k}$  are called similarly. Substituting Eqs.(30) and (31) into  $f$ , we obtain

$$f(\vec{k}\vec{x} + \phi, \vec{k}) = f(\vec{k}^R\vec{x} + \phi^R, \vec{k}^R) + \epsilon[(\delta \vec{k}\vec{x}^R + \delta \vec{k}\vec{X} + \delta \phi)\frac{\partial f}{\partial \theta} + \delta \vec{k}\frac{\partial f}{\partial \vec{k}}]. \quad (32)$$

Comparing this with Eq.(29), we obtain the cancellation condition of the singularities as

$$\delta \theta + X_{lmn}\rho_{lmn}^{(3)} + X_{lm}\rho_{lm}^{(2)} + X_l\rho_l^{(1)} = 0, \quad (33)$$

$$\delta k_n + 3X_{lm}\rho_{lmn}^{(3)} + 2X_l\rho_{ln}^{(2)} = 0, \quad (34)$$

where we put

$$\delta \theta = \delta \vec{k}\vec{X} + \delta \phi. \quad (35)$$

As a result, the renormalized perturbation series becomes

$$\begin{aligned} w &= f(\vec{k}^R(\vec{X}, t)\vec{x} + \phi^R(\vec{X}, t), \vec{k}^R(\vec{X}, t)) \\ &+ \epsilon\{[x_{lmn}^R\rho_{lmn}^{(3)} + x_{lm}^R\rho_{lm}^{(2)} + x_l^R(\rho_l^{(1)} + \delta k_l)]\frac{\partial f}{\partial \theta} \\ &+ (3x_{lm}^R\rho_{lmn}^{(3)} + 2x_l^R\rho_{ln}^{(2)})\frac{\partial f}{\partial k_n} + a^{(0)}\}. \end{aligned} \quad (36)$$

It seems strange that a singular term  $\delta k_l$  remains in the renormalized perturbation series. However, since it appears with the form  $x_l^R\delta k_l$  and this term simply narrow the valid region of the result to  $|x_l^R\delta k_l| \sim O(1)$ . In this way, the renormalizability of the singular behavior has been shown, and our perturbation result is meaningful *locally* around the arbitrary position  $\vec{X}$ . Thus, obviously, we can put  $\vec{x} = \vec{X}$  and then obtain

$$w(\vec{X}, t) = f(\vec{k}^R(\vec{X}, t)\vec{X} + \phi^R(\vec{X}, t), \vec{k}^R(\vec{X}, t)) + \epsilon a^{(0)}, \quad (37)$$

where note that  $\vec{X}$  can be replaced by a small letter  $\vec{x}$  because  $\vec{X}$  is arbitrary. This is a *globally* valid expression for  $w$ . Here,  $\vec{k}^R(\vec{X}, t)$  and  $\phi^R(\vec{X}, t)$  are given by Eqs.(30) and

(31) with Eqs.(33) -(35). Before discussing these equations, we consider a relation between  $\vec{k}^R(\vec{X}, t)$  and  $\theta^R(\vec{X}, t)$  which is defined by

$$\theta^R(\vec{X}, t) = \vec{k}^R(\vec{X}, t)\vec{X} + \phi^R(\vec{X}, t). \quad (38)$$

By using  $\theta^R$ ,  $\theta = \vec{k}\vec{x} + \phi$  is written as

$$\theta = \vec{k}^R\vec{x}^R + \theta^R + \epsilon(\delta\vec{k}\vec{x}^R + \delta\theta). \quad (39)$$

Differentiating this equation with respect to  $\vec{X}$ , we obtain

$$\vec{k}^R = \frac{\partial\theta^R}{\partial\vec{X}}, \quad (40)$$

where we have used the equalities  $\partial\theta/\partial X_l = \partial k_l/\partial X_m = 0$  and

$$\delta\vec{k} = \frac{\partial\delta\theta}{\partial\vec{X}}. \quad (41)$$

The last equality can be easily proved from Eqs.(33) and (34). One may find that our  $\theta^R$  corresponds to  $\Theta$  in the Cross-Newell's formulation for the phase dynamics [15]. Actually, in the next section, by the renormalization group analysis, we will derive the equation for  $\theta^R(\vec{x}, t)$ , which corresponds to a Cross-Newell phase equation.

## V. RENORMALIZATION GROUP

Noting  $\vec{k}(t) = \vec{k}^R(\vec{0}, t)$ , we rewrite Eq.(31) as

$$\vec{k}^R(\vec{0}, t) = \vec{k}^R(\vec{X}, t) + \epsilon\delta\vec{k}(\vec{X}, t). \quad (42)$$

This equation is valid locally around  $\vec{X} = \vec{0}$  because of the singular behavior of  $\delta\vec{k}$ . (See Eq.(34).) Here, the capital letter  $\vec{X}$  in the argument of renormalized quantities is convention.  $\vec{x}$  can be used instead of it, of course. We now wish to obtain a globally valid expression for  $\vec{k}^R$ . The RG analysis makes us possible to do it. We first show a formal argument. Differentiating Eq.(42) with respect to  $\vec{X}$ , we obtain

$$\frac{\partial k_l^R}{\partial X_m} = -\epsilon \frac{\partial \delta k_l}{\partial X_m}, \quad (43)$$

$$= \epsilon(6X_n \rho_{lmn}^{(3)} + 2\rho_{lm}^{(2)}). \quad (44)$$

Here,  $\rho^{(3)}$  and  $\rho^{(2)}$  are functions of  $\vec{k}(t)$  in their definitions Eqs.(25) and (26), but we are allowed to replace  $\vec{k}(t)$  to  $\vec{k}^R(\vec{X}, t)$  because the difference gives the  $O(\epsilon^2)$  contribution in Eq.(44). Multiplying  $D_{lm}(\vec{k}^R)$  to this equation and using Eqs.(25) and (26), we obtain

$$D_{lm}(\vec{k}^R) \frac{\partial k_l^R}{\partial X_m} = \epsilon \left[ \frac{\partial k_l^R}{\partial t} X_l + \frac{\partial \phi^R}{\partial t} \right], \quad (45)$$

$$= \epsilon \frac{\partial \theta^R}{\partial t}. \quad (46)$$

By combining Eq.(40), this equation can be written as

$$\epsilon \frac{\partial \theta^R}{\partial t} = D_{lm}(\vec{k}^R) \frac{\partial}{\partial X_l} \frac{\partial}{\partial X_m} \theta^R. \quad (47)$$

We expect that Eq.(47) is globally valid because it does not include a singularity. Equation (47) is equivalent to a Cross-Newell equation [15]. However, one may claim that Eq.(47) is valid only over the region where Eq.(42) is meaningful because Eq.(47) has been derived from Eq.(42). Also, the replacement from  $\vec{k}$  to  $\vec{k}^R$  in Eq.(44) seems a little bit unnatural. We notice here that the RG analysis is not employed explicitly in the above argument. Next, we consider the problem from the RG point of view.

The argument in this paragraph follows a review article by Shirkov [17]. The RG is a symmetry structure with respect to the alternation of the way of giving boundary values [17–19]. Let us see this property in our system.  $Q$  denotes a set of renormalized quantities, i.e.

$$Q(\vec{X}) = (\theta^R(\vec{X}, *), \vec{k}^R(\vec{X}, *)), \quad (48)$$

where  $Q(\vec{X})$  is a function in  $t$ . From Eq.(42) and a similar expression for  $\theta^R$  derived from Eq.(39), we can write

$$Q(\vec{0}) = Q(\vec{X}) + \epsilon \delta Q(\vec{X}). \quad (49)$$

This expression is valid locally around  $\vec{X} = \vec{0}$ . Using the spatial homogeneity of the system, we can prove

$$Q(\vec{X}_0) = Q(\vec{X}) + \epsilon \delta Q(\vec{X} - \vec{X}_0), \quad (50)$$

which is valid locally around an arbitrary position  $\vec{X}_0$ . Then, from the form of Eq.(50), we assume a globally valid expression

$$Q(\vec{X}) = F(\vec{X} - \vec{X}_0, Q(\vec{X}_0)) \quad (51)$$

for an arbitrary position  $\vec{X}_0$ . Since  $Q(\vec{X})$  does not depend on  $\vec{X}_0$ , Eq.(51) shows the invariance property of  $Q(\vec{X})$  under the alternation of the way of giving the boundary values. Further,  $F$  satisfies the transitivity expressed by

$$F(\vec{X} - \vec{X}_0, Q(\vec{X}_0)) = F(\vec{X} - \vec{X}_1, F(\vec{X}_1 - \vec{X}_0, Q(\vec{X}_0))). \quad (52)$$

This leads to the composition law of the transformation  $R(\vec{X})$  acting on  $Q(\vec{X}_0)$ , which is defined by

$$R(\vec{X})Q(\vec{X}_0) = F(\vec{X}, Q(\vec{X}_0)). \quad (53)$$

Noting that  $R(\vec{0}) = 1$  and that  $R(-\vec{X})$  is the inverse transformation of  $R(\vec{X})$ , we can conclude that the transformations form a Lie group. Then, according to the fundamental theorem of the Lie group theory [20], a differential equation obtained by considering an infinitely small transformation determines the Lie group. The differential equation is

$$\frac{\partial Q}{\partial \vec{X}} = \vec{\beta}(Q), \quad (54)$$

where  $\vec{\beta}$  is called a generator and defined by

$$\vec{\beta}(Q) = \left. \frac{\partial F(\vec{X}, Q)}{\partial \vec{X}} \right|_{\vec{X}=\vec{0}}. \quad (55)$$

Eqs.(52) and (54) are called functional and differential RG equations for the variable  $Q$  respectively. The generator  $\vec{\beta}(Q)$  has been referred to as a beta function in the RG literature.

As seen in its definition, the generator is determined by the local property of the transformation. Thus, the locally valid expression Eq.(49) is enough to give the generator perturbatively, and through the RG equation Eq.(54), which is globally valid, we can find  $Q(\vec{X})$  for all  $\vec{X}$ . This procedure to obtain globally improved solutions from locally valid ones are known as the perturbative RG method. From Eqs.(33) and (34), we obtain

$$\frac{\partial \theta^R}{\partial X_l} = \beta_l^{(\theta)}(\theta^R, \vec{k}^R) = k_l^R, \quad (56)$$

$$\frac{\partial k_l^R}{\partial X_m} = \beta_{lm}^{(\vec{k})}(\theta^R, \vec{k}^R) = \frac{1}{2}(D^{-1}(\vec{k}^R))_{lm} \frac{\partial \theta^R}{\partial t}. \quad (57)$$

These RG equations are equivalent to Eq.(47). The procedures Eqs.(43)-(46) should be regarded as a simplified argument of the perturbative RG method based on the Lie group theory. Recently, Kunihiro has claimed that the RG method developed by Goldenfeld, Oono and their collaborators is not mathematical and has proposed an envelop method against the RG formulation [21]. However, there is no necessity to consider the envelop method, though it may be a correct interpretation of the perturbative RG method. We recognize that the RG method discussed here is not purely mathematical, but still remains at a formal level as usual in theoretical physics. We believe however that a careful mathematical description will be possible. (See [22] for a mathematical formulation for phase dynamics.)

## VI. DISCUSSION

We summarize the result: The solution is expressed in the form

$$w = f(\theta^R, \vec{k}^R) + \epsilon a^{(0)}, \quad (58)$$

where  $\theta^R$  obeys the nonlinear phase equation Eq.(47), and  $\vec{k}^R$  is derived from Eq.(40). As far as we know, the general formula for phase diffusion coefficients has never been presented explicitly. When a model equation satisfies the conditions we assumed, we can immediately calculate the phase diffusion coefficients by using the general formula Eq.(27). In Appendix B, we show the result for the SH equation. Since many variants of the SH eq. [1] satisfy

the conditions we assumed, this general formula is useful. However, there are some important classes out of our consideration. One class consists of nonlocal models which appear commonly in fluid dynamics because of the incompressible condition [23]. The resultant phase equation is coupled to mean flow and this leads to a new type of instability called a skewed varicose instability [15,24,25]. It seems that there is no technical difficulty to discuss nonlocal models, but complication will come in. The other class out of our consideration consists of models which have a different type of a family of solutions. As one example, since the Kuramoto-Sivashinsky (KS) equation [26] possesses a Galilei symmetry [27], a family of spatially periodic solutions in the one dimensional system is expressed by  $w = f(k(x - vt) + \phi, k) + v$ , where  $k$ ,  $\phi$  and  $v$  are arbitrary constants. Correspondingly, due to the existence of additional null mode, secular terms take a different form when we employ a naive perturbative expansion. We expect that all secular divergences are renormalized to  $k$ ,  $\phi$  and  $v$  and the RG equation gives a correct phase equation. We remark that an oscillatory instability appearing in Rayleigh-Bénard convection with a low Prandtl number is associated with the Galilei invariance [28].

A pattern breaks the spatial translational symmetry and the phase variable is interpreted as a Goldstone mode for the symmetry breaking. Similarly, phase equations associated with the temporal translation symmetry have been discussed in reaction diffusion systems [3], where a limit cycle solution breaks the temporal translational symmetry. The resultant phase equation has a similar form with a Burgers equation. Renormalization group derivation of the phase equations from general reaction diffusion equations which have a limit cycle solution is much simpler than the derivation in this paper [29], because the naive perturbation result in such systems shows that there is only one secular term proportional to  $t$ . Phase equations for propagating patterns will be derived similarly, though a little complication arises because of the mixed nature of spatial and temporal symmetry breakings.

Let us compare the RG method with other methods for perturbative system reduction. The most efficient derivation of phase equations may be the Cross-Newell method. In their formulation, besides a standard multiple-scales analysis, a new variable  $\Theta$  is introduced

with playing two roles: the derivative of  $\Theta$  gives the modulation of wavevectors, while  $\Theta$  has a multiple-scales relation with a phase coordinate specifying a position of the basic periodic pattern. That is, their formulation consists of a tactical combination of the multiple-scales analysis and the method of changing variables. We recognize that such an acrobatic procedure greatly reduces a calculation time to obtain a final form especially in complicated problems. Their method may be optimized for the purpose of deriving nonlinear phase equations. Also, the formal perturbation series can be produced systematically up to an arbitrary order.

The RG derivation of phase equations need more tedious calculation than the Cross-Newell method. As far as we are concerned with simple problems such as nonlinear oscillations, there is no significant difference between computational efficiencies by the RG derivation and by conventional ones. However, as shown in this paper, it seems obvious that a more mechanical method requires a more calculation time when the problem becomes complicated. Thus, one may doubt the practical efficiency of mechanical formulation. Mechanical formulation seems to be apparently impractical. However, for much complicated problems which are intractable by manual calculation, a more mechanical method may become more useful because mechanical procedures can be programmed in computers. Further, mechanical methods are useful to problems for which we cannot find a suitable guess of the solution.

We should remark that our derivation is not yet a completely mechanical one. As discussed in section II, there is subtle ambiguity where a small parameter  $\epsilon$  is put. We cannot formalize this procedure mechanically. This problem is not peculiar to the present problem, but appears in many examples, even in a quite simple example such as linear ordinary differential equations [11]. It may be honest to say that we set up systems so that the RG method works well. Our choice is physically reasonable, but our purpose is to eliminate such a phrase. In addition, let us recall that perturbative system reduction is possible only when we can express the system in question by introducing a small parameter. The formulation on this part is independent of the perturbation method and should be discussed separately.

However, we do not have a general way how to express what we wish to describe. Of course, in some cases, we can introduce suitably a small parameter with referring to the physical situation in consideration, but again our purpose is to eliminate such a phrase. As far as we consider in the present knowledge, it seems hopeless to formulate this point mechanically.

As mentioned in section I, there are other theories aiming at a mechanical formulation. Bogaevsky and Povzner analyzed many examples including WKB-type problems and some PDE problems [7]. The correspondence with the RG method seems clear at least in simple problems. Their method is interpreted as systematic renormalized perturbation theory, that is, they consider renormalized variables from the outset. The calculation may be carried out mechanically, but we do not check it whether or not their formulation has the same content as the RG method even for complicated problems such as phase equations derived in this paper.

A Kuramoto's geometrical interpretation on the system reduction comes from the notion of a normal form [8]. In this sense, his formulation shares a common structure with Bogaevsky and Povzner. However, his main claim is that the system reduction is independent of choice of the method of expansion series. One may choose any expansion method as far as it can be performed consistently. In general, we can obtain a formal solution to a set of self-consistent equations under certain additional assumptions which are required from the outset in his formulation. According to his view, when the perturbative system reduction is considered, this formal solution is evaluated with referring to physical situations. This is one clear strategy, but seems less practical than standard conventional methods and less mechanical than the RG method.

The importance of the RG method is not restricted to its mechanical nature. As is well known, RG has been used in diverse fields of theoretical physics in particular since Wilson formulated RG in a different form [30]. Thus, by studying a common structure to several RG approaches, we may find a new method for system reduction.

Shirkov reviewed that RG methods appeared in several contexts were employed based on the transitive property of physically relevant quantities with respect to the alternation of



the way of giving the boundary values [17]. He called this property functional self-similarity (FSS). The notion is almost equivalent to a group structure of the old style RG [19,18] and is a generalization of the usual self-similarity related to power laws. In this sense, the RG analysis is interpreted as a method to construct a FSS symmetric solution. It is not easy to know the FSS symmetry beforehand. Nevertheless, if we find it before calculation, the group analysis [31] can be employed to an enlarged system consisting of the original equation in question and an equation describing alteration of the boundary data. Such a formulation was already proposed by Kovalev, Krisvenko and Pustovalov [32].

The RG method employed in this paper is identical to the old style RG developed in 1950'. Another popular formulation of RG is the Wilson-Kadanoff scheme, in which the RG transformation is devised based on a physical insight of the system [30,33]. Owing to its constructive nature, such a scheme may be useful in non-perturbative system reduction. The study in this direction is developing [34]. Also, related to it, the constructive RG method was employed to discuss asymptotic behavior of PDEs mathematically [35]. However, we do not yet understand a relation among these different RG approaches.

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## VII. APPENDIX A: CALCULATION OF SECULAR TERMS

In this appendix, we solve Eq.(23) with Eqs.(20)-(22). We consider the case  $M = 4$  which holds for the SH eq. The extension to arbitrary  $M(> 4)$  is straightforward, and the

result is unchanged.

The equations in question are regarded as linear functional equations for  $2\pi$  periodic functions in  $\theta = \vec{k}\vec{x} + \phi$ . In order to solve it, we consider an Hilbert space  $V_{per} = \{u | u \in L_2[0, 2\pi], u(0) = u(2\pi)\}$ , where  $\partial/\partial x_j$  in  $L_{1:n}^{(n)}$  is replaced by  $k_j \partial/\partial \theta$  when the operator  $L_{1:n}^{(n)}$  acts on  $u \in V_{per}$ . Before solving the equations, we summarize elementary mathematical notions about linear algebra on the space  $V_{per}$ .

There exists a null eigenvector  $\Phi_0$  satisfying

$$\hat{L}^{(0)}\Phi_0 = 0. \quad (59)$$

In fact, differentiating the equality  $F|_{w=f} = 0$  with respect to  $\theta$ , we obtain

$$\hat{L}^{(0)}\frac{\partial f}{\partial \theta} = 0. \quad (60)$$

Thus, we can put

$$\Phi_0 = \partial f / \partial \theta. \quad (61)$$

This null eigenvector is associated to the existence of a family of solutions parameterized with  $\phi$  which comes from a spatially translational invariance of the system.

A generalized null space in  $V_{per}$  is spanned by only  $\Phi_0$  from the following two reasons. First, the other null mode which is associated with a family of solutions with parameterized by  $\vec{k}$  is not included in the space  $V_{per}$ . Second, if there were another null mode, a family of solutions would be expressed in a different form. (For a related discussion, see section V.) Let us discuss the first point. Differentiating the equality  $F|_{w=f} = 0$  with respect to  $\vec{k}$ , we obtain

$$\hat{L}^{(0)}\frac{df}{d\vec{k}} = \hat{L}^{(0)}\left(\vec{x}\frac{\partial f}{\partial \theta} + \frac{\partial f}{\partial \vec{k}}\right) = 0. \quad (62)$$

Then, it is easily found that the null mode  $\vec{x}\partial f/\partial \theta + \partial f/\partial \vec{k}$  is not a periodic function in  $\vec{x}$ .

The inner product in the space  $V_{per}$ , which is denoted by  $(u, v)$ , is defined by

$$(u, v) = \frac{1}{2\pi} \int_0^{2\pi} d\theta u(\theta)v(\theta). \quad (63)$$

$\hat{L}^{(0)}$  is not an Hermite operator in general. (Note that  $L^{(0)}$  is an Hermite operator for the SH eq.) Then, an adjoint null eigenvector is defined by

$$\hat{L}^{(0)\dagger}\Phi_0^\dagger = 0, \quad (64)$$

where  $\Phi_0^\dagger$  is assumed to be normalized as

$$(\Phi_0^\dagger, \Phi_0) = 1. \quad (65)$$

Consider a linear equation for  $u$  in the space  $V_{per}$

$$\hat{L}^{(0)}u = b, \quad (66)$$

where  $b$  is a  $2\pi$  periodic function. Since  $\hat{L}^{(0)}$  has a null eigenvector, there is a solution only under the solvability condition

$$(\Phi_0^\dagger, b) = 0. \quad (67)$$

When this condition is satisfied, the solution  $u$  to Eq.(66) is expressed by

$$u = \hat{M}b + \lambda\Phi_0, \quad (68)$$

where  $\hat{M}$  is a pseudo-inverse operator of  $\hat{L}^{(0)}$  and  $\lambda$  is an arbitrary constant. ( $\hat{M}\hat{L}^{(0)} = \hat{L}^{(0)}\hat{M} = 1$  holds on a restricted space which consists of  $u$  satisfying  $(\Phi_0^\dagger, u) = 0$ .)

Here, for the later convenience, we write down an identity

$$-\hat{M}\hat{L}_j^{(1)}\Phi_0 = \frac{\partial f}{\partial k_j}, \quad (69)$$

which is obtained by expressing Eq.(62) in the form

$$\hat{L}_j^{(1)}\Phi_0 + \hat{L}^{(0)}\frac{\partial f}{\partial k_j} = 0. \quad (70)$$

Now, we derive Eqs.(24) - (27). We first show  $a^{(n)} = 0$  for  $n \geq 4$ . We rewrite Eq.(23) with Eq.(20) explicitly as

$$\hat{L}^{(0)} a_{(1:5)}^{(5)} = 0, \quad (71)$$

$$5\hat{L}_{(5:5)}^{(1)} a_{(1:5)}^{(5)} + \hat{L}^{(0)} a_{(1:4)}^{(4)} = 0, \quad (72)$$

$$10\hat{L}_{(4:5)}^{(2)} a_{(1:5)}^{(5)} + 4\hat{L}_{(4:4)}^{(1)} a_{(1:4)}^{(4)} + \hat{L}^{(0)} a_{(1:3)}^{(3)} = 0, \quad (73)$$

$$10\hat{L}_{(3:5)}^{(3)} a_{(1:5)}^{(5)} + 6\hat{L}_{(3:4)}^{(2)} a_{(1:4)}^{(4)} + 3\hat{L}_{(3:3)}^{(1)} a_{(1:3)}^{(3)} + \hat{L}^{(0)} a_{(1:2)}^{(2)} = 0, \quad (74)$$

From Eq.(71), we obtain

$$a_{1:5}^{(5)} = \rho_{1:5}^{(5)} \Phi_0, \quad (75)$$

where  $\rho_{1:5}^{(5)}$  is a constant whose value will be determined later. We substitute Eq.(75) to Eq.(72). Then, the solvability condition for  $a^{(4)}$  is satisfied due to the parity symmetry, and we obtain  $a^{(4)}$ :

$$a_{1:4}^{(4)} = \rho_{1:4}^{(4)} \Phi_0 + 5\rho_{1:5}^{(5)} \frac{\partial f}{\partial k_{i_5}}, \quad (76)$$

where we have used Eq.(69), and  $\rho_{1:4}^{(4)}$  is a constant whose value will be determined later. We substitute Eqs.(75) and (76) to Eq.(73). Then, the solvability condition for  $a^{(3)}$  yields

$$\rho_{1:5}^{(5)} D_{4:5} = 0, \quad (77)$$

where

$$D_{mn} = \frac{1}{2}(\Phi_0^\dagger, \hat{L}_{mn}^{(2)} \Phi_0) + (\Phi_0^\dagger, \hat{L}_m^{(1)} \frac{\partial f}{\partial k_n}). \quad (78)$$

Using the regularity of the matrix  $D$  which is assumed, we obtain

$$\rho_{1:5}^{(5)} = 0. \quad (79)$$

Under the solvability condition, we obtain  $a^{(3)}$ :

$$a_{1:3}^{(3)} = \rho_{1:3}^{(3)} \Phi_0 + 4\rho_{1:4}^{(4)} \frac{\partial f}{\partial k_{i_4}}, \quad (80)$$

where we have used Eq.(69), and  $\rho_{1:3}^{(3)}$  is a constant whose value will be determined later. We substitute Eqs.(75)-(80) to Eq.(74). Repeating the same argument, we obtain

$$\rho_{1:4}^{(4)} = 0, \quad (81)$$

and

$$a_{1:2}^{(2)} = \rho_{1:2}^{(2)} \Phi_0 + 3\rho_{1:3}^{(3)} \frac{\partial f}{\partial k_{i_3}}. \quad (82)$$

One can easily see that  $a^{(n)} = 0$  ( $n \geq 4$ ) even for the case  $M > 4$ . Then, Eq.(23) with Eqs.(21) and (22) becomes

$$3\hat{L}_{(2:3)}^{(2)} a_{(1:3)}^{(3)} + 2\hat{L}_{(2:2)}^{(1)} a_{(1:2)}^{(2)} + \hat{L}^{(0)} a_{(1:1)}^{(1)} = \frac{\partial k_{i_1}}{\partial t} \frac{\partial f}{\partial \theta}, \quad (83)$$

$$\hat{L}_{(1:3)}^{(3)} a_{(1:3)}^{(3)} + \hat{L}_{(1:2)}^{(2)} a_{(1:2)}^{(2)} + \hat{L}_{(1:1)}^{(1)} a_{(1:1)}^{(1)} + \hat{L}^{(0)} a^{(0)} = \frac{\partial k_{i_1}}{\partial t} \frac{\partial f}{\partial k_l} + \frac{\partial \phi}{\partial t} \frac{\partial f}{\partial \theta} \quad (84)$$

The solvability condition for  $a^{(1)}$  yields

$$6\rho_{1:3}^{(3)} D_{2:3} = \frac{\partial k_{i_1}}{\partial t} \quad (85)$$

Under the solvability condition, we obtain  $a^{(1)}$ :

$$a_{1:1}^{(1)} = \rho_{1:1}^{(1)} \Phi_0 + 2\rho_{1:2}^{(2)} \frac{\partial f}{\partial k_{i_2}}, \quad (86)$$

where we have used the parity symmetry and Eq.(69).  $\rho_{1:1}^{(1)}$  is a constant whose value is *not* determined. Repeating the same argument for Eq.(84), we obtain

$$2\rho_{1:2}^{(2)} D_{1:2} = \frac{\partial \phi}{\partial t}. \quad (87)$$

$a^{(0)}$  is unnecessary for our argument, though we can calculate it.

## VIII. APPENDIX B: PHASE DIFFUSION COEFFICIENTS

In this appendix, we derive concrete expressions of phase diffusion coefficients for the SH eq. based on the general formula Eq.(27). A spatially periodic solution of the SH eq. is expressed by

$$w(x) = f(\theta, \vec{k}) = A_1(k) \cos \theta + A_3(k) \cos \theta + \dots, \quad (88)$$

where  $\theta = \vec{k}\vec{x} + \phi$ , and

$$A_1 = \frac{2}{\sqrt{3}}\sqrt{R - (1 - k^2)^2}. \quad (89)$$

Since  $A_3 \sim O(R^{3/2})$  for  $R \rightarrow 0$ , we consider only one mode. Hereafter, we will not use the smallness of  $R$ , but keep in mind that the validity is restricted to the small  $R$ . If someone wishes to obtain phase diffusion coefficients for large  $R$ , he needs to write a computer program. However, main features of the phase diffusion coefficients can be obtained under this assumption. Then,  $\Phi_0$ ,  $\partial f / \partial k_l$ , and  $\Phi_0^\dagger$  are easily given by

$$\Phi_0 = -A_1 \sin \theta, \quad (90)$$

$$\frac{\partial f}{\partial k_l} = \frac{2(1 - k^2)k_l}{R - (1 - k^2)^2} A_1 \cos \theta, \quad (91)$$

$$\Phi_0^\dagger = -\frac{1}{\pi A_1} \sin \theta. \quad (92)$$

Substituting these expressions and  $\{\hat{L}^{(n)}\}_{n=0}^2$  given by Eqs.(15) and (17) to the general formula Eq.(27), we obtain

$$D_{nm} = -2(1 - k^2)\delta_{nm} + 4k_n k_m - \frac{8(1 - k^2)^2 k_n k_m}{R - (1 - k^2)^2}. \quad (93)$$

Since the system has a rotational symmetry,  $D_{nm}$  is further written as

$$D_{nm} = D_{//}(k^2) \frac{k_n k_m}{k^2} + D_{\perp}(k^2) (\delta_{nm} - \frac{k_n k_m}{k^2}), \quad (94)$$

where

$$D_{//} = 6k^2 - 2 - \frac{8(1 - k^2)^2 k^2}{R - (1 - k^2)^2}, \quad (95)$$

$$D_{\perp} = -2(1 - k^2). \quad (96)$$

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